A NOTE ON GROWTH OF FOURIER TRANSFORMS AND MODULI OF CONTINUITY ON DAMEK RICCI SPACES

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Abstract. We obtain results related to boundedness of the growth of Fourier transform by the modulus of continuity on Damek-Ricci spaces. For noncompact riemannian symmetric spaces of rank one, analogues of all the results follow the same way.

1. Introduction

This article is motivated by a recent paper of Bray and Pinsky ([3]) on growth properties of Fourier transform on euclidean spaces and their analogues on noncompact riemannian symmetric spaces of rank one. The two main results for symmetric spaces X proved in [3] are the following (for notation see [3] and Section 2):

Theorem 1.1. *Let $p \in [1, 2]$ and $f \in L^p(X)$. Then for $|\eta| < \gamma_p$, $\lambda \in \mathbb{R}$ and $r \geq r_0 > 0$*

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \int_K |\widetilde{f}(\lambda + i\eta, k)| dk \right] \leq C_{p,\eta} \Omega_p[f] \left(\frac{1}{r}\right).$$

Theorem 1.2. *For $p \in (1, 2)$ let f be a K-finite function in $L^p(X)$. Then for $|\eta| < \gamma_p \rho$, $\lambda \in \mathbb{R}$ and $r \geq r_0 > 0$*

$$\left(\int_{\mathbb{R}} \min\left\{1, \left(\frac{\lambda}{r}\right)^{2p'}\right\} \int_K |\widetilde{f}(\lambda + i\eta, n)|^{p'} dk |c(\lambda)|^{-2} d\lambda \right)^{1/p'} \leq C_{p,\eta} \Omega_p[f] \left(\frac{1}{r}\right).$$

The natural question whether this theorem can be generalized without putting the restriction of K-finiteness is asked by the authors in [3]. They have also conjectured the existence of analogues of these results for radial functions on harmonic NA groups, which are also known as Damek-Ricci spaces. We shall use both of these names.

We recall that a riemannian symmetric space X of rank one is a quotient space G/K where G is a connected noncompact semisimple Lie group of real rank one with finite centre and K is a maximal compact subgroup of G. We also recall that a harmonic NA group S is a solvable Lie group. The distinguished prototypes of them are the noncompact riemannian symmetric spaces of rank one, which account for a very small subclass of the class of all NA groups (see [1]). All the results proved in this article for NA groups will have natural interpretations on symmetric spaces and proving them will be simpler. Some of the intrinsic difficulties of working with NA groups is the lack of G-action and in particular K-action (in other words lack of symmetry), the noncompactness of the subgroup N which somewhat takes the place of the maximal compact group K and lack of a rich representation theoretic background.

Purpose of this article is to consider the theorems mentioned above for general functions on Damek-Ricci spaces and to put them in a form which reveals their differences with that on the euclidean spaces.

2000 *Mathematics Subject Classification.* Primary 43A85; Secondary 22E30.

Key words and phrases. harmonic NA groups, moduli of continuity, spherical mean.

This work is supported by a research grant no. 48/1/2006-R&DII/1488 of National Board for Higher Mathematics, India.

Two main ingredients of the proof of these theorems in [3] are analogues of Hausdorff-Young inequality (for radial and K-finite functions proved in [7, 8]) and *restriction* theorem (proved in [13, 15]); precisely, for $1 \leq p < 2$ and $|\eta| < (2/p-1)\rho$

$$|\widetilde{f}(\lambda+i\eta,k)|dk \leq C\|f\|_p.$$

However, it turns out that on symmetric spaces or more generally on NA groups one can have stronger analogues of restriction theorem (see [14] and Section 3). In the case of symmetric spaces these results can be interpreted as norm estimates of certain matrix coefficients of class-1 principal series representations of the underlying group G and can be linked to the Kunze-Stein phenomenon (see [4]). We need to elaborate on this. Like radial functions the Fourier transform of a general L^p-function also exist on the strip S_p (defined in Section 2) parallel to the real line. But there is an *angular* variable (k for symmetric space and n for NA group). If f is a radial function in $L^p(X)$ then $\sup_\lambda |\widehat{f}(\lambda)| \leq C\|f\|_p$ in a smaller strip. But in the absence of radiality one is concerned about the behavior in k variable of the Fourier transform. We see that the behavior depends on $\Im\lambda$. That is inside the strip on every line parallel to the real axis Fourier transform behaves differently in k variable. We may stress that in particular it changes as we move from $\alpha+i\eta$ to $\alpha-i\eta$. In the context of symmetric space this can be attributed to Herz's principe de majoration) and Kunze-Stein phenomenon ([4, 5]).

Coming to the second theorem (which is an application of Hausdorff-Young inequality) we notice that, unlike $\mathbb{R}^n$ on NA the following well known inequality is not true:

$$\sup_{\lambda\in\mathbb{R}} \|\widetilde{f}(\lambda,\cdot)\|_{L^\infty(K)} \leq C\|f\|_1.$$

In fact the best result we know is:

$$\sup_{\lambda\in\mathbb{R}} \|\widetilde{f}(\lambda,\cdot)\|_{L^2(K)} \leq C\|f\|_1.$$

This indicates that on these spaces the Hausdorff-Young inequality will involve *mixed norms* which will change as λ will vary over a strip on the complex plane. Strips which are symmetric about the real line are the natural domains of the Fourier transforms $\widetilde{f}(\lambda,k)$ for various Lebesgue and Lorentz spaces. We may stress here that finer subdivisions of L^p spaces called Lorentz spaces appear naturally in this set up for both of these theorems (see Section 3 for details).

Such results are recently proved by the authors ([14]). In this article we further improve these theorems and as consequences obtain new analogues of Theorem 1.1 and 1.2. We conclude by noticing that the L^p-norm of the spherical mean operator M_t defined in [3] decays exponentially as $t \to \infty$ (proved in Section 4 and Section 6). This is a noneuclidean phenomenon which also vindicates that in these spaces results will be different from those on the euclidean set up.

2. PRELIMINARIES

Most of the preliminaries can be found in [2, 1, 14]. To make the article self-contained we shall gather only those results which are required for this paper. For a detailed account we refer to [14].

Everywhere in this article for any $p \in [1,\infty)$, $p' = p/(p-1)$ and $\gamma_p = 2/p-1$, $\gamma_\infty = -1$. We note that $\gamma_p = -\gamma_{p'}$. For a complex number z, we will use $\Re z$ and $\Im z$ to denote respectively the real and imaginary parts of z. For $p \in [1,2]$ we define

$$S_p = \{z \in \mathbb{C} \mid |\Im z| \leq \gamma_p\rho\}.$$

By S_p° we denote the interior of the strip.

We will follow the standard practice of using the letter C for constant, whose value may change from one line to another. Occasionally the constant C will be suffixed to show its dependency on important parameters. The letters $\mathbb{C}$ and $\mathbb{R}$ will denote the set of complex and real numbers respectively.

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a H-type algebra where $\mathfrak{v}$ and $\mathfrak{z}$ are vector spaces over $\mathbb{R}$ of dimensions m and k respectively. Indeed $\mathfrak{z}$ is the centre of $\mathfrak{n}$ and $\mathfrak{v}$ is its ortho-complement with respect to the inner product of $\mathfrak{n}$. Let $N = \exp\mathfrak{n}$. We shall identify $\mathfrak{v}$ and $\mathfrak{z}$ and N with $\mathbb{R}^m$, $\mathbb{R}^k$ and $\mathbb{R}^m\times\mathbb{R}^k$ respectively. Elements of A

will be identified with $a_t = e^t, t \in \mathbb{R}$. A acts on N by nonisotropic dilation: $\delta_{a_t}(X,Y) = (e^{-t}X, e^{-2t}Y)$. Let $S = NA$ be the semidirect product of N and A under the action above. Then S is a solvable connected and simply connected Lie group with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$. It is well known that S is a nonunimodular amenable Lie group. The homogenous dimension of S is $Q = m/2 + k$. For convenience we shall also use the symbol ρ for $Q/2$. An element $x = na = n(X,Y)a \in S$ can be written as (X,Y,a), $X \in \mathfrak{v}, Y \in \mathfrak{z}, a \in A$. Precisely (X,Y,a) is identified with $\exp(X+Y)a$. We shall use the notation $A(x) = A(na_t) = t$.

A function f on S is called *radial* if for all $x, y \in S$, $f(x) = f(y)$ if $d(x,e) = d(y,e)$, where d is the metric induced by the canonical left invariant riemannian structure of S. For a radial function f we shall also use $f(t)$ to mean $f(a_t)$.

For a suitable function f on S its radialization Rf is defined as

$$Rf(x) = \int_{S\nu} f(y)d\sigma_\nu(y), \tag{2.1}$$

where $\nu = r(x)$ and $d\sigma_\nu$ is the surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S \mid d(y,e) = \nu\}$ normalized by $\int_{S_\nu} d\sigma_\nu(y) = 1$. It is clear that Rf is a radial function and if f is radial then $Rf = f$.

The Poisson kernel $\mathcal{P} : S \times N \longrightarrow \mathbb{R}$ is given by $\mathcal{P}(na_t, n_1) = P_{a_t}(n_1^{-1}n)$ where

$$P_{a_t}(n) = P_{a_t}(V,Z) = Ca_t^Q\left(\left(a_t + \frac{|V|^2}{4}\right)^2 + |Z|^2\right)^{-Q}, \; n = (V,Z) \in N. \tag{2.2}$$

The value of C is adjusted so that $\int_N P_a(n)dn = 1$ and $P_1(n) \le 1$ (see [2, (2.6)]). We also need the following:

(1) $P_a(n) = P_a(n^{-1})$.
(2) $P_{a_t}(n) = P_1(a_{-t}na_t)e^{-2\rho t}$.
(3) $\mathcal{P}(x,n) = \mathcal{P}(n_1a_t, n) = P_{a_t}(n^{-1}n_1) = P_{a_t}(n_1^{-1}n)$.
(4) $\mathcal{P}_\lambda(x,n) = \mathcal{P}(x,n)^{1/2 - i\lambda/Q} = \mathcal{P}(x,n)^{-(i\lambda-\rho)/Q}$
(5) $R(\mathcal{P}_\lambda(\cdot,n))(x) = \phi_\lambda(x)\mathcal{P}_\lambda(e,n)$, $R(e^{(i\lambda-\rho)A(\cdot)})(x) = \phi_\lambda(x)$.

The action of class-1 principal series representation $\pi_\lambda, \lambda \in \mathbb{C}$ realized on functions on N is given by:

$$(\pi_{-\lambda}(n_1a_t)\phi)(n) = \phi(a_{-t}n_1^{-1}na_t)e^{t(i\lambda-\rho)}.$$

From this it is easy to verify that $(\pi_{-\lambda}(x)P_1^{1/2-i\lambda/Q})(n) = \mathcal{P}_\lambda(x,n)$.

The elementary spherical function $\phi_\lambda(x)$ is given by

$$\phi_\lambda(x) = \langle \pi_\lambda(x)P_1^{1/2-i\lambda/Q}, P_1^{1/2-i\lambda/Q}\rangle_{L^2(N)} = \int_N \mathcal{P}_\lambda(x,n)\mathcal{P}_{-\lambda}(e,n)dn.$$

It follows that ϕ_λ is a radial eigenfunction of the Laplace-Beltrami operator $\mathcal{L}$ of S with eigenvalue $-(\lambda^2 + \rho^2)$ satisfying $\phi_\lambda(x) = \phi_{-\lambda}(x), \phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. As $\mathcal{P}_{-i\rho}(x,n) \equiv 1$ for all $x \in S$ and $n \in N$ and $\mathcal{P}_{i\rho}(x,n) = \mathcal{P}(x,n)$,

$$\phi_{-i\rho}(x) = \int_N \mathcal{P}_{i\rho}(e,n) = \int_N P_1(n)dn = 1.$$

For $\alpha = \frac{m+k-1}{2}$ and $\beta = \frac{k-1}{2}$, ϕ_λ is identical with the Jacobi function $\phi_\lambda^{(\alpha,\beta)}$ with the *ideal situation* of $\alpha > \beta > -\frac{1}{2}$ (see[1]). Thus spherical Fourier transform is related to the Jacobi transform.

We define the spherical Fourier transform $\widehat{f}$ of a suitable radial function f as

$$\widehat{f}(\lambda) = \int_S f(x)\phi_\lambda(x)dx,$$

whenever the integral converges.

The left invariant Haar measure on S decomposes as

$$\int_S f(x)dx = \int_{N\times A} f(na_t)e^{-2\rho t}dtdn,$$

where $dn(X,Y) = dX\,dY$ and dX, dY, dt are Lebesgue measures on $\mathfrak{v}$, $\mathfrak{z}$ and $\mathbb{R}$ respectively.

Jacobians of the following transformations will be required for our computations.

(a) $\int_N f(a_t n a_{-t}) = \int_N f(n)e^{-2\rho t}dn$.

(b) $\int_S R_y f(x)dx = \int_S f(xy)dx = \int_S f(x)dx e^{2\rho A(y)}$, i.e. the modular function $\Delta(y) = e^{-2\rho A(y)}$. Here R_y denotes the right-translation operator.

(c) $\int_S f(x^{-1})dx = \int_S f(x)e^{2\rho A(x)}dx$ and $\int_S f(x^{-1})e^{2\rho A(x)}dx = \int_S f(x)dx$.

For two measurable functions f and g on S we define their convolution as (see [9, p. 51]):

$$f * g(x) = \int_S f(y)g(y^{-1}x)dy = \int_S f(y^{-1})g(yx)\Delta(y^{-1})dy = \int_S f(xy^{-1})g(y)\Delta(y^{-1})dy.$$

For a measurable function f on S we define its Fourier transform (which is an analogue of the Helgason Fourier transform on the symmetric space) by

$$\widetilde{f}(\lambda, n) = \int_S f(x)\mathcal{P}_\lambda(x,n)dx,$$

whenever the integral converges. If f is radial then using (5) above we see that $\widetilde{f}(\lambda,n) = \widehat{f}(\lambda)\mathcal{P}_\lambda(e,n)$.

The Poisson transform of a function F on N is defined as (see [2])

$$\mathfrak{P}_\lambda F(x) = \int_N F(n)\mathcal{P}_\lambda(x,n)dn.$$

Any norm estimate involving the Fourier transform of a function is equivalent to a dual statement involving the Poisson transform. Precisely, for a function f on S, a function F on N and for $\lambda \in \mathbb{C}$,

$$\|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)} \leq C\|f\|_p \iff \|\mathfrak{P}_\lambda F\|_{L^{p'}(S)} \leq C\|F\|_{L^{q'}(N)}.$$

We shall denote the (p,q)-Lorentz spaces by $L^{p,q}(S)$ and the corresponding norm by $\|\cdot\|^*_{p,q}$. We recall that $L^{p,\infty}(S)$ is called weak L^p-space. For definitions and other details on Lorentz spaces we refer to [10, 16, 14].

3. Existence and some properties of the Fourier transform

The following two theorems are proved by the authors in [14].

Theorem 3.1. *Let f be a measurable function in the Lorentz space $L^{p,q}(S)$.*

(i) *If $1 \leq p < 2$ and $q = 1$ then there exists a subset N^p of N of full Haar measure, depending only on f, such that $\widetilde{f}(\lambda,n)$ exists for all $n \in N^p$ and $\lambda \in S_p$.*

(ii) *If $1 < p < 2$ and $1 < q \leq \infty$ then there exists a subset N^p of N of full Haar measure, depending only on f, such that $\widetilde{f}(\lambda,n)$ exists for all $n \in N^p$ and $\lambda \in S_p^\circ$.*

(iii) *If p,q are as in* (ii) *then there exists a subset N'_p of N of full Haar measure, depending only on f, such that $\widetilde{f}(\lambda,n)$ exists for all $n \in N'_p$ and almost every $\lambda \in \partial S_p$.*

Theorem 3.2 (Riemann-Lebesgue Lemma)**.** *Let $1 \leq p < 2$. If $f \in L^{p,1}(S)$ then for almost every fixed $n \in N$ the map $\lambda \mapsto \widetilde{f}(\lambda,n)$ is continuous on S_p and analytic on S_p°. Furthermore*

$$\lim_{|\xi|\to\infty} \widetilde{f}(\xi + i\eta, n) = 0$$

uniformly in $\eta \in [-\gamma_p\rho, \gamma_p\rho]$.

For functions in $L^{p,q}(S)$, $q > 1$ the assertions above remain valid for $\lambda \in S_p^\circ$ and for $\eta \in [-(\gamma_p\rho - \delta), (\gamma_{p_1}\rho - \delta)]$ for any $0 < \delta < \gamma_p$.

Here are improved versions of some relevant theorems proved in [14]:

Theorem 3.3 (Restriction on line). *For $f \in L^1(S)$, $q \in [1,\infty]$ and $\alpha \in \mathbb{R}$,*

$$\left(\int_N |\widetilde{f}(\alpha + i\gamma_q\rho, n)|^q dn\right)^{1/q} \leq \|f\|_1.$$

For $f \in L^{p,\infty}(S), 1 < p < 2$, $p < q < p'$ and $\alpha \in \mathbb{R}$,

$$\left(\int_N |\widetilde{f}(\alpha + i\gamma_q\rho, n)|^q dn\right)^{1/q} \leq C_{p,q}\|f\|^*_{p,\infty}.$$

Proof. We will prove only the second part. We take $p_1, p_2 \geq 1$ such that $p_1 < p < p_2 < q < p'$. Using the result in [14, Theorem 4.2] we have

$$\|\widetilde{f}(\alpha + i\gamma_q\rho, \cdot)\|_{L^q(N)} \leq C_{p_1,q}\|f\|_{p_1}$$

which is equivalent to the following by duality:

$$\|\mathfrak{P}_{\alpha+i\gamma_q\rho}\xi\|_{p_1'} \leq C_{p_1,q}\|\xi\|_{L^{q'}(N)}.$$

Through similar arguments we also get

$$\|\mathfrak{P}_{\alpha+i\gamma_q\rho}\xi\|_{p_2'} \leq C_{p_2,q}\|\xi\|_{L^{q'}(N)}.$$

We interpolate between the two results above ([10, p. 64, 1.4.2]) to get

$$\|\mathfrak{P}_{\alpha+i\gamma_q\rho}\xi\|^*_{p',1} \leq C_{p_1,p_2,p,q}\|\xi\|_{L^{q'}(N)}.$$

as $p_2' < p' < p_1'$. The last result is equivalent to (by duality)

$$\|\widetilde{f}(\alpha + i\gamma_q\rho, \cdot)\|_{L^q(N)} \leq C_{p_1,p_2,p,q}\|f\|^*_{p,\infty}.$$

□

It is clear that for p, q, α as in the second part of the theorem above and $1 \leq r < \infty$, if $f \in L^{p,r}(S)$ then

$$\left(\int_N |\widetilde{f}(\alpha + i\gamma_q\rho, n)|^q dn\right)^{1/q} \leq C_{p,q}\|f\|^*_{p,r}.$$

To have a norm estimate of $\widetilde{f}(\lambda, \cdot)$ which is uniform over the strip S_q, we consider a weighted measure space $(N, P_1(n)dn)$.

Corollary 3.4 (Restriction on strip). *Let*

$$L^q(N, P_1) = \{f \text{ measurable on } N \mid \int_N |f(n)|^q P_1(n)dn < \infty\}.$$

(a) *Let $1 \leq p < q \leq 2$ and $1 \leq r \leq q$. If $f \in L^{p,\infty}(S)$ then*

$$\|\widetilde{f}(\lambda, \cdot)\|_{L^r(N,P_1)} \leq C_{p,q}\|f\|^*_{p,\infty}$$

for any λ in the strip $\mathcal{S}_q = \{z \in \mathbb{C} \mid |\Im z| \leq \gamma_q\rho\}$.

(b) *Let $1 \leq p < q < 2$ and $f \in L^{p,\infty}(S)$. Then*

$$\|\widetilde{f}(\lambda, \cdot)\|_{L^{q,1}(N,P_1)} \leq C_{\lambda,p,q}\|f\|^*_{p,\infty} \text{ for all } \lambda \in S_q^\circ.$$

(c) *For $p < q < q_1 \leq 2$, $\lambda \in \mathbb{R}$*

$$\|\widetilde{f}(\lambda + i\gamma_{q_1}, \cdot)\|^*_{L^{q,1}(N,P_1)} \leq C_{p,q,q_1}\|f\|^*_{p,\infty}$$

Proof. Applying the arguments of [14, Corollary 4.4] on Theorem 3.3 we get (a).

For (b) we take a $\lambda \in S_q^\circ$. Then $\lambda = \alpha + i\gamma_{q_1}\rho$ for some $q < q_1 \leq 2$. We choose a q_2 such that $1 \leq q_2 < q < q_1 \leq 2$. By Theorem 3.3 and as $P_1(n) \leq 1$

$$\|\widetilde{f}(\alpha + i\gamma_{q_1}\rho, \cdot)\|_{L^{q_1}(N,P_1)} \leq C_{p,q_1}\|f\|^*_{p,\infty}.$$

As $(N, P_1(n))$ is a finite measure space, this implies

$$\|\widetilde{f}(\alpha + i\gamma_{q_1}\rho, \cdot)\|_{L^{q_2}(N,P_1)} \leq C_{p,q_1}\|f\|^*_{p,\infty}.$$

An interpolation ([10, p.64]) between these two results gives (b).

In (c) we make the constant independent of λ by fixing q_1. □

We consider the product measure space $(Y, dy) = (N, dn) \times (\mathbb{R}, |c(\lambda)|^{-2}d\lambda)$. For a measurable function $F(\lambda, n)$ on this measure space we denote mixed norm of F by

$$\|F\|^*_{(q;p',r)} = \left\| \left(\int_N |F(\cdot, n)|^q dn \right)^{1/q} \right\|^*_{p',r}.$$

Theorem 3.5 (Hausdorff-Young Theorem)**.** *Let* $1 \leq p \leq 2$. *Then*

(a) *for* $p \leq q \leq p'$,

$$\left(\int_{\mathbb{R}} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q\rho, n)|^q dn \right)^{\frac{p'}{q}} |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{p'}} \leq C_{p,q}\|f\|_p.$$

(b) *for* $p < q < p'$, $p' \leq r \leq \infty$ *and* $1 \leq s \leq \infty$

$$\|\widetilde{f}(\cdot + i\gamma_q, \cdot)\|^*_{(q;r,s)} \leq C_{p,q,r}\|f\|^*_{p,s}.$$

Proof. Part (a) is proved in [14, Theorem 4.6].

For (b) we consider the operator T between the measure spaces (S, dx) and $(\mathbb{R}, |(c(\lambda)|^{-2}d\lambda)$ defined by:

$$Tf(\lambda) = \|\widetilde{f}(\lambda + i\gamma_q\rho, \cdot)\|_q.$$

We choose p_1 and p_2 such that $1 \leq p_1 < p < p_2 < q$. Then by [14, Corollary 4.7] $\|Tf\|_{p_1'} \leq C_{p,q}\|f\|_{p_1}$ and $\|Tf\|_{p_2'} \leq C_{p,q}\|f\|_{p_2}$. Interpolating them ([16, p.197])) we get (b). □

We note that part (b) of the theorem above generalizes an euclidean result (see [16, p. 200]) where teh inequality is proved for $r = s = p'$.

4. The spherical mean operators

Let σ_t be the normalized surface measure of the geodesic sphere of radius t. For a suitable function f on S we define the spherical mean operator $M_t f = f * \sigma_t$. Using the radialization operator R (see Section 2) then

$$M_t f(x) = R({}^x f)(t)$$

where ${}^x f$ is the right-translation of f by x.

We need to make the radialization operator more precise. We note that if $d(x, e) = t$ where $x = na_r \in S$ and $n = (X, Y)$ then

$$(\cosh t)^2 = \left[\cosh r + e^r |X|^2\right]^2 + e^{2r}|Y|^2.$$

We define the surface for any $t \geq |s|$,

$$T_{t,s} = \{(X, Y) \in \mathbb{R}^m \times \mathbb{R}^k \mid (\cosh t)^2 = \left[\cosh r + e^r |X|^2\right]^2 + e^{2r}|Y|^2\}.$$

Then $T_{t,s}$ is the set of points $P = P(X,Y) \in \mathbb{R}^m \times \mathbb{R}^k = N$ such that $d(Pa_s, e) = t$. Let $dw_{t,s}$ be the induced measure on $T_{t,s}$ such that for a suitable function Φ on $\mathbb{R}^m \times \mathbb{R}^k$,

$$\int_{\mathbb{R}^m\times\mathbb{R}^k} \Phi(X,Y)dXdY = \int_{t\geq|s|}\left[\int_{T_{t,s}} \Phi(P)dw_{t,s}(P)\right]dt.$$

Then the radialization opertaor R can be defined by the following:

$$R(\Phi)(t) = \int_{|s|<t}\left[\int_{T_{t,s}} \Phi(X,Y,a_s)dw_{t,s}(X,Y)\right]e^{-2\rho s}ds.$$

Using this expression of radialization we shall prove the (p,p) property of M_t.

Proposition 4.1. *For* $1 \leq p \leq \infty$

$$\|M_t f\|_p \leq \phi_{i\gamma_p\rho}(a_t)\|f\|_p.$$

Proof. For convenience let us denote the variable point (X,Y,a_s) in the integration defining radialization R simply by P. We recall that $M_t(f)(x) = R({}^x f)(t)$.

$$\begin{aligned}
&\left(\int_S |M_t f(x)|^p dx\right)^{1/p}\\
=\;& \left(\int_S \left|\int_{|s|<t}\int_{T_{t,s}} {}^x f(P)dw_{t,s}\, e^{-2\rho s}ds\right|^p dx\right)^{1/p}\\
\leq\;& \int_{|s|<t}\int_{T_{t,s}}\left(\int_S |{}^x f(P)|^p dx\right)^{1/p} dw_{t,s}\, e^{-2\rho s}ds\\
=\;& \int_{|s|<t}\int_{T_{t,s}}\left(\int_S |f(x)|^p e^{2\rho s}dx\right)^{1/p} dw_{t,s}\, e^{-2\rho s}ds\\
=\;& \|f\|_p \int_{|s|<t}\int_{T_{t,s}} e^{2\rho s/p}\, dw_{t,s}\, e^{-2\rho s}ds\\
=\;& \|f\|_p\, \phi_{i\gamma_p\rho}(a_t).
\end{aligned}$$

In the last step we have used that $e^{2\rho s/p} = e^{2\rho A(P)/p}$ and as $1/p = (1/2 - i(i\gamma_p\rho)/2\rho)$, $R(e^{2\rho A(\cdot)/p})(t) = \phi_{i\gamma_p\rho}(a_t)$ (see Section 2). ☐

Since for $t > 0$ $\phi_{i\gamma_p\rho}(a_t) \asymp e^{-(2\rho/p')t}$ for $1 \leq p \leq 2$ and $\phi_{i\gamma_{p'}\rho} = \phi_{i\gamma_p\rho}$, we have from above $\|M_t f\|_{op} \leq e^{-(2\rho/p')t}$ or $\leq e^{-(2\rho/p)t}$ depending on $p \leq 2$ or $p > 2$. Here $\|M_t f\|_{op}$ is the operator norm of M_t from $L^p(S)$ to $L^p(S)$. The proof of the proposition above for symmetric space is given in Section 6.

Using interpolation ([16, p.197]) we have

$$\|M_t f\|^*_{p,s} \leq C_{p,s}\|f\|^*_{p,s}$$

for $p \in (1,\infty)$, $s \in [1,\infty]$.

Proposition 4.2. *For $f \in L^1(S)$ $M_t f$ converges to f in L^1 as $t \to 0$. Also for all $f \in L^{p,q}(S), 1 < p < \infty$, $1 \leq q \leq \infty$ $M_t f$ converges to f in L^p as $t \to 0$.*

A standard argument involving dominated convergence theorem and approximation by functions in $C_c^\infty(S)$ proves the result for L^p-spaces. If $f \in L^{p,q}(S)$ with p,q as above, then there exists $f_1 \in L^1(S), f_2 \in L^r(S)$ with $r \in (p,2]$ such that $f = f_1 + f_2$ (see [14]). Use of this decomposition gives the result for Lorentz spaces.

Proposition 4.3. *For $f \in L^p(S), 1 \leq p < 2$, $(M_t f)\widetilde{\ }(\lambda,n) = \widetilde{f}(\lambda,n)\phi_\lambda(t)$.*

Proof. We note that

$$\begin{aligned}\int_S f(xy)\mathcal{P}_\lambda(x,n_1)dx &= \int_S f(xy)(\pi_{-\lambda}(x)P_1^{1/2-i\lambda/Q})(n_1)dx\\ &= \int_S f(z)(\pi_{-\lambda}(zy^{-1})P_1^{1/2-i\lambda/Q})(n_1)e^{2\rho A(y)}dz\\ &= \int_S f(z)(\pi_{-\lambda}(z)(\pi_{-\lambda}(y^{-1})P_1^{1/2-i\lambda/Q}))(n_1)dz\, e^{2\rho A(y)}\\ &= (\pi_{-\lambda}(f)\mathcal{P}_\lambda(y^{-1},\cdot))(n_1)e^{2\rho A(y)}. \qquad (4.1)\end{aligned}$$

In the computations below we shall use the following substitutions in different steps: $y = na_s$ and $x = n_2a_r$, $n_3 = a_{-s}n^{-1}a_s$, $n' = a_{-r}n_2^{-1}n_1a_r$. We shall also use the fact that $R\mathcal{P}_\lambda(\cdot,n)(t) = \mathcal{P}_\lambda(e,n)\phi_\lambda(t)$ (see section 2).

$$\begin{aligned}(M_tf)\widetilde{\ }(\lambda,n_1)) &= \int_S M_tf(x)\mathcal{P}_\lambda(x,n_1)dx\\ &= \int_{|s|\le t}\int_{T_{t,s}}\int_S f(xy)\mathcal{P}_\lambda(x,n_1)dxdw_{t,s}(n)e^{-2\rho s}ds\\ &= \int_{|s|\le t}\int_{T_{t,s}}\left[(\pi_{-\lambda}(f)\mathcal{P}_\lambda(y^{-1},\cdot))(n_1)e^{2\rho s}\right]dw_{t,s}(n)e^{-2\rho s}ds\\ &= \int_{|s|\le t}\int_{T_{t,s}}\left[\int_S f(x)(\pi_{-\lambda}(x)\mathcal{P}_\lambda(y^{-1},\cdot))(n_1)e^{2\rho s}\,dx\right]dw_{t,s}(n)e^{-2\rho s}ds\\ &= \int_{|s|\le t}\int_{T_{t,s}}\left[\int_{N\times\mathbb{R}} f(n_2a_r)\mathcal{P}_\lambda(y^{-1}a_{-r}n_2^{-1}n_1a_r)e^{r(i\lambda-\rho)}e^{2\rho s}dx\right]dw_{t,s}(n)e^{-2\rho s}ds\\ &= \int_S f(x)\left[\int_{|s|\le t}\int_{T_{t,s}}\mathcal{P}_\lambda(y^{-1}n')e^{2\rho s}dw_{t,s}(n)e^{-2\rho s}ds\right]e^{r(i\lambda-\rho)}\,e^{-2\rho r}dn_2\,dr\\ &= \int_{N\times\mathbb{R}} f(n_2a_r)\left[\int_{|s|\le t}\int_{T_{t,s}}\mathcal{P}_\lambda(n_3a_{-s},n')e^{4\rho s}dw_{t,s}(n_3)e^{-2\rho s}ds\right]e^{r(i\lambda-\rho)}\,e^{-2\rho r}dn_2\,dr\\ &= \int_{N\times\mathbb{R}} f(n_2a_r)\left[R(\mathcal{P}_\lambda(\cdot,n'))(t)\right]e^{r(i\lambda-\rho)}\,e^{-2\rho r}dn_2\,dr\\ &= \int_{N\times\mathbb{R}} f(n_2a_r)\mathcal{P}_\lambda(e,n')\phi_\lambda(t)e^{r(i\lambda-\rho)}\,e^{-2\rho r}dn_2\,dr\\ &= \int_{N\times\mathbb{R}} f(n_2a_r)(\pi_{-\lambda}(x)P_1^{1/2-i\lambda/Q})(n_1)dx\phi_\lambda(t)\ \text{(by 4.1)}\\ &= \widetilde{f}(\lambda,n_1)\phi_\lambda(t).\end{aligned}$$

□

Following Bray we define *spherical modulus of continuity* for any $1 \le p,q \le \infty$ as

$$\Omega_{p,q}[f](r) = \sup_{0<t\le r}\|M_tf - f\|^*_{p,q}.$$

We note that $\Omega_{p,p}$ is the same as Ω_p. We shall quote two lemmas from [3]. Let j_α be the usual Bessel function of the first kind normalized by $j_\alpha(0) = 1$.

Lemma 4.4. *For* $\alpha > -1/2$

$$C_{1,\alpha}\min\left\{1,\left(\frac{\lambda}{r}\right)^2\right\} \le \int_0^1\left(1 - j_\alpha\left(\frac{\lambda z}{r}\right)\right)dz \le \sup_{0\le z\le 1}\left(1 - j_\alpha\left(\frac{\lambda z}{r}\right)\right) \le C_{2,\alpha}\min\left\{1,\left(\frac{\lambda}{r}\right)^2\right\}$$

for two positive constants $C_{1,\alpha}$ and $C_{2,\alpha}$.

Lemma 4.5. *Let $\alpha \geq \beta \geq -1/2$, $t_0 > 0$ and $|\eta| \leq \rho$. Then for all $0 \leq t \leq t_0$,*

$$|1 - \phi^{(\alpha,\beta)}_{\mu+i\eta}(a_t)| \geq C|1 - j_\alpha(\mu t)|$$

for some positive constant $C = C_{t_0,\alpha,\beta}$. Consequently

$$\int_0^1 \left|1 - \phi^{(\alpha,\beta)}_{\mu+i\eta}\left(\frac{z}{r}\right)\right| dz \geq C \min\left\{1, \left(\frac{\mu}{r}\right)^2\right\}.$$

5. Growth of Fourier transform and Moduli of continuity

We offer the following modification of two main Theorems in [3] mentioned in the introduction.

Theorem 5.1. *Let $r \geq r_0 > 0$ be fixed.*
(a) *Let $q \in [1, \infty]$. Then for $f \in L^1(S)$ and $\lambda \in \mathbb{R}$*

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q\rho, n)|^q dn\right)^{1/q}\right] \leq C_q \Omega_1[f]\left(\frac{1}{r}\right).$$

(b) *Let $1 < p < 2, p < q < p'$. Then for $f \in L^{p,\infty}(S)$ and $\lambda \in \mathbb{R}$*

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q\rho, n)|^q dn\right)^{1/q}\right] \leq C_{p,q} \Omega_{p,\infty}[f]\left(\frac{1}{r}\right).$$

(c) *Let $1 \leq p < q \leq 2$ and $f \in L^{p,\infty}(S)$. Then for $|\eta| < \gamma_p \rho$ and $\lambda \in \mathbb{R}$*

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \left(\int_N |\widetilde{f}(\lambda + i\eta, n)|^q P_1(n) dn\right)^{1/q}\right] \leq C_{p,q} \Omega_{p,\infty}[f]\left(\frac{1}{r}\right).$$

The proofs follow from Theorem 3.3, Corollary 3.4 (a) and the arguments of the corresponding result in [3]. We omit it for brevity. One can also prove similar results using part (b) and (c) of Corollary 3.4.

We need the following lemma.

Lemma 5.2. *Let $1 < p \leq 2$ and $1 \leq q \leq \infty$ Let g be a nonnegative bounded continuous function on $[0,1] \times S$ and f be a nonnegative function in $L^{p,q}(S)$. Then,*

$$\left\| \left(\int_0^1 g(t, \cdot)) dt\right) f \right\|^*_{p,q} \leq \sup_{[0,1]} \|g(t, \cdot) f\|^*_{p,q}.$$

The proof uses a standard argument involving duality and Fubini's theorem.

Theorem 5.3. (a) *Let $1 \leq p \leq 2$ and $p \leq q \leq p'$. Then for $f \in L^p(S)$*

$$\left(\int_{\mathbb{R}} \min\left\{1, \left(\frac{\lambda}{r}\right)^{2p'}\right\} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q\rho, n)|^q dn\right)^{p'/q} |c(\lambda)|^{-2} d\lambda\right) \leq C_{p,q} \Omega_p[f]\left(\frac{1}{r}\right).$$

(b) *Let $1 < p \leq 2$, $p < q < p'$ and $1 \leq s \leq \infty$. Then for $f \in L^{p,s}(S)$ and $\alpha \in [p', \infty]$,*

$$\left\| \min\left\{1, \frac{\cdot}{r}\right\}^2 \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|^*_{\alpha,s} \leq C_{p,q,r,s} \Omega_{p,s}[f]\left(\frac{1}{r}\right).$$

Proof. We shall prove only (b). (b) We apply Theorem 3.5 (b) on the function $M_tf - f \in L^{p,s}(S)$ and subsequently use Lemma 4.5, Lemma 4.4 and Lemma 5.2 to get the result through the following steps.

$$\left\| |\phi_{\cdot+i\gamma_q\rho}(a_t) - 1| \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|_{\alpha,s}^* \leq C_{p,q,r,s}\|M_tf - f\|_{p,s}^*.$$

Form this we get

$$\sup_{0\leq t\leq 1/r} \left\| |\phi_{\cdot+i\gamma_q\rho}(a_t) - 1| \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|_{\alpha,s}^* \leq C_{p,q,r,s}\Omega_{p,s}[f]\left(\frac{1}{r}\right).$$

This implies

$$\sup_{0\leq z\leq 1} \left\| |1 - j_\alpha(\cdot\frac{z}{r})| \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|_{\alpha,s}^* \leq C_{p,q,r,s}\Omega_{p,s}[f]\left(\frac{1}{r}\right).$$

From this using Lemma 5.2 we get

$$\left\| \left(\int_0^1 |1 - j_\alpha(\cdot\frac{z}{r})| dz\right) \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|_{\alpha,s}^* \leq C_{p,q,r,s}\Omega_{p,s}[f]\left(\frac{1}{r}\right)$$

which implies

$$\left\| \min\left\{1, \left(\frac{\cdot}{r}\right)^2\right\} \left(\int_N |\widetilde{f}(\cdot + i\gamma_q\rho, n)|^q dn\right)^{1/q} \right\|_{\alpha,s}^* \leq C_{p,q,r,s}\Omega_{p,s}[f]\left(\frac{1}{r}\right).$$

Through similar arguments we can prove (a) applying Theorem 3.5 (a) on the function $M_tf - f$ □

6. Appendix

In this section we consider the noncompact riemannian symmetric spaces X of rank one. Most of the notations are standard and can be found in [3]. It is not difficult to see that all the theorems proved for Damek-Ricci spaces will have analogue for symmetric spaces where N will be replaced by K and the Fourier transform defined in Section 2 will be substituted by the usual Helgason Fourier transform. As K is compact and hence a finite measure space, some of the statements will look simpler here, e.g. $P_1(n)$ will be substituted by 1. We shall omit these results here except for one additional result for symmetric spaces (Theorem 6.3). We begin with a proof of the norm estimate of M_t.

For a function f on G/K, let $M_tfx) = \int_K f(xka_t)dk$. Then M_tf is a right K-invariant function and hence a function on G/K. We will see below that M_t is a bounded operator from $L^p(G/K)$ to $L^p(G/K)$ for every $p \geq 1$ and $\|M_t\|_{op} \leq e^{-(2\rho/p')t}$ or $e^{-(2\rho/p)t}$ depending on $p \leq 2$ or $p > 2$. Here $\|\cdot\|_{op}$ denotes the operator norm.

Proposition 6.1. *M_t is strong type (p,p) and $\|M_tf\|_p \leq \|f\|_p\phi_{i\gamma_p\rho}(a_t)$.*

Proof. If $x = nak$ then $M_tf(x) = M_tf(na)$. Thus $\int_G |M_tf(x)|^p dx = \int_{N\times\mathbb{R}} |M_tf(n_2a_s)|^p e^{-2\rho s} dn_2 ds$. Therefore

$$(6.1) \qquad \|M_tf\|_p \leq \int_K \left(\int_X |f(xka_t)|^p dx\right)^{1/p} dk = \int_K \left(\int_{N\times\mathbb{R}} |f(n_2a_ska_t)|^p e^{-2\rho s} dn_2 ds\right)^{1/p} dk.$$

For the inside integral we put $ka_t = n_1 a_r k_1$. (Then $H(a_t^{-1}k^{-1}) = -r$.)

$$\begin{aligned}
&\int_{N\times\mathbb{R}} |f(n_2 a_s k a_t)|^p e^{-2\rho s} dn_2 ds \\
= &\int_{N\times\mathbb{R}} |f(n_2 a_s n_1 a_r)|^p e^{-2\rho s} dn_2 ds \\
= &\int_{N\times\mathbb{R}} |f(n_2 n_3 a_{s+r})|^p e^{-2\rho s} dn_2 ds \text{ where } n_3 = a_s n_1 a_{-s} \\
= &\int_{N\times\mathbb{R}} |f(n a_s)|^p e^{-2\rho s} e^{2\rho r} dn ds \\
= &\ \|f\|_p^p e^{2\rho r} = \|f\|_p^p e^{-2\rho H(a_t^{-1}k^{-1})}.
\end{aligned}$$

We put this back in (6.1) to get $\|M_t f\|_p \leq \|f\|_p \,\phi_{i\gamma_p\rho}(a_t)$. □

The proposition above has the following interesting corollary.

Corollary 6.2. *For any $p \in [1,2)$ if $\lambda \in S_p^\circ$ then $|\phi_\lambda(a_t)| \leq \phi_{i\gamma_p\rho}(a_t)$.*

Note that there is no constant in the inequality.

Proof. We note that $M_t\phi_\lambda(x) = \int_K \phi_\lambda(xka_t)dk = \phi_\lambda(x)\phi_\lambda(a_t)$. For $p \in [1,2)$, we take $\lambda \in \mathcal{S}_p^\circ$. Then $\phi_\lambda \in L^{p'}(S)$. Using the Proposition above we see that, $\|M_t\phi_\lambda\|_{p'} \leq \phi_{i\gamma_p}(a_t)\|\phi_\lambda\|_{p'}$ and hence $|\phi_\lambda(a_t)|\|\phi_\lambda\|_{p'} \leq \phi_{i\gamma_p}(a_t)\|\phi_\lambda\|_{p'}$. Thus $|\phi_\lambda(a_t)| \leq \phi_{i\gamma_p}(a_t)$ for all $\lambda \in S_p^\circ$. □

On rank one symmetric space $X = G/K$ by [12, Lemma 1] we have the following additional result.

Theorem 6.3. *If $1 < p < 2$, then*

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \left(\int_K |\widetilde{f}(\lambda + i\gamma_p\rho, k)|^p dk\right)^{1/p}\right] \leq C_p \Omega_{p,1}[f]\left(\frac{1}{r}\right)$$

and

$$\sup_\lambda \left[\min\left\{1, \left(\frac{\lambda}{r}\right)^2\right\} \left(\int_K |\widetilde{f}(\lambda - i\gamma_p\rho, k)|^{p'} dk\right)^{1/p'}\right] \leq C_p \Omega_{p,1}[f]\left(\frac{1}{r}\right).$$

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